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The Nature of Prospective Mathematics Teachers' Designed Manipulatives and their Potential as Anchors for Conceptual and Pedagogical Knowledge

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Denish Akuom & Steven Greenstein

Montclair State University, USA

Abstract: While traditionally teachers have been positioned as implementers of curricular materials designed by others, this work positions them as designers of their own curricular resources, thereby inviting opportunities for their exploration at the intersection of content, pedagogy, and design. As researchers accepting greater responsibility for preparing teachers to maintain a commitment to their pedagogical vision in practice, this work seeks to cultivate the imagination of humanistic forms of mathematics teaching and learning by supporting these explorations. Toward that end, this paper reports on research that examines connections between the pedagogical/conceptual knowledge that prospective teachers embed in the designs of original manipulatives and how those designs mediate the pedagogical moves they make in teaching situations. The promise of this work is in connections that may offer a viable means to support bolder connections between teacher preparation and practice. We share findings from the analysis of prospective teachers' design activity that conveys (1) the diversity of design decisions, rationales, and mediating resources that it entailed, and (2) how the designed manipulative act as anchors for their conceptual/pedagogical moves. The implications of these findings for teacher preparation and professional learning are considered.

Keywords: Making, Teacher Knowledge, Technology, Preservice Teacher Education, 3D Designing

Introduction

Cai and Cirillo (2014) assert that "if [theoretical knowledge] is to be useful, it ultimately must be translated to practice" (p. 138). This is the ultimate aim of the research we report here, to offer implications that connect theory and practice in order to provide practical solutions to the perennial problem that teachers often experience considerable challenges in transferring the knowledge they construct in teacher preparation into their practice (Ünver, 2014; Spillane & Zeuli, 1999). Although the National Academy of Education (2005) found that teacher education programs that link teacher preparation coursework to field experiences tend to be more effective than those that do not, many colleges and universities implement teacher education programs with coursework entirely situated in inauthentic education settings. According to Kazemi et al. (2009), although future teachers tend to craft their pedagogies as they learn about research-supported instructional methods, teacher educators also stress the importance of developing one's practice in real classrooms with real students. With this critical concern in mind, this work seeks to determine the means by which teachers can transform their knowledge from theory into practice through approximations of practice (Grossman et al., 2009) that simulate the work of teaching.

Traditionally, teachers have been positioned as implementers of curricular resources designed by others. While doing this may seem convenient, it comes with the potential to collapse the space of possibilities for how teaching and learning might occur. And with increasing access to digital design and fabrication technologies around the world, we believe this may elicit new opportunities that may challenge this presumption and disrupt the consequent denial of agency that it entails. Thus, the framing of *teachers as* *designers* is an orientation that may now be embraced more fully than ever. In the study reported here, we explore the 'life' of teachers as designers (e.g., Kalantzis & Cope, 2010; Maher, 1987; Svihla et al., 2015) of their own curricular materials, tracing their design activities from *tool-design* to *tool-use*.

We broadly conceive design to include the "intentional activity of transforming ideas and knowledge" (Carvalho et al., 2019, p. 79) into "tangible, meaningful artifacts" (Koehler & Mishra, 2005, p. 135). Our purpose in doing so is to present a novel *Making* experience within mathematics teacher preparation that we hypothesized would inform their curricular and pedagogical thinking and cultivate images of themselves as agents of curricular and pedagogical reform (Leander & Osborne, 2008; Priestley et al., 2012). *Making*, in this sense, is conceived as the creative production of artifacts via activities that include designing, building, and innovating with tools and materials to solve practical problems (Halverson & Sheridan, 2014). Thus, the experience tasks prospective mathematics teachers (PMTs) with digitally designing, 3D printing, and evaluating original manipulatives that are responsive to the curricular (Dewey, 1990; Pinar et al., 1995) needs and interests of actual learners.

While there is a considerable body of research on *students'* mathematical *Making* (e.g., Bower et al., 2020; Valente & Blikstein, 2019), research is only beginning to uncover the benefits that teachers experience in *Making* contexts (Akuom & Greenstein, 2021; Akuom et al., to appear; Greenstein et al., 2017, 2018, 2019, 2020, 2021, 2021, to appear). Research has yet to explore the conceptual, pedagogical, social, cultural, and experiential resources that PMTs bring to their design decisions, the rationales for their uses of those resources, and how these resources and rationales intersect to mediate those design decisions. Additionally, there is still a need for further research to identify viable means for connecting the pedagogical/conceptual knowledge that prospective teachers construct in teacher preparation with their eventual teaching practice. This work speaks to these theoretical and practical gaps by addressing two questions: *As prospective teachers Make new manipulatives for mathematics teaching and learning, (1) What is the nature of the resources and rationales they bring to their design decisions and how do these intersect to mediate their decision making? (2) Can connections be made between pedagogical/conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?* If connections can be made between the knowledge that prospective teachers construct in teacher preparation, how that knowledge materializes in their designs of physical manipulatives, and how those knowledge-embedded designs mediate their teaching interactions, we propose that these findings can illuminate and subsequently strengthen the relationship between instructional intention and enactment, in particular (see Remillard, 2018), and teacher preparation and practice more broadly.

Theoretical Framework

As this work traces PMTs' activities from designing a manipulative to evaluating its use in practice, two theoretical lenses are used to respond to these questions. We organize them next according to the phases of PMTs' activity in which they were employed.

Design Phase

In relation to what PMTs' might learn through the design phase of their activity, we organized our theoretical framing around the theories of constructivism and constructionism. These theories recognize that knowledge is actively constructed by a learner (Piaget, 1970; von Glasersfeld, 1995), with constructionism adding the dimension that the knowledge be constructed through the process of making a shareable object (Harel & Papert, 1991) within a collaborative social context. In order to characterize the interplay between a designer's knowledge, experiences, intentions, and other resources as they are invoked during the iterative design of the shareable object, we appealed to Koehler and Mishra's (2005) *Learning by Design* approach, which engages participants in the activity of designing – or the purposeful imagining, planning, and intending that interacts with Making – by calling on them to "actively engage in inquiry, research and design" so that they can make "tangible, meaningful artifacts" that represent "the end products of the learning process" (p. 135). In our case, in the context, these end products are the manipulatives that prospective teachers will share with children with the intention of promoting their mathematical learning. And as PMTs design them, it is their intention (Malafouris, 2013) to embed them with particular affordances (Gibson, 1977) for utilization schemes (Verillon & Rabardel, 1995) that they hypothesize will enable the child to abstract, through their sensorimotor engagement (Kamii & Housman, 2000; Piaget, 1970), the perceptual elements that are the basis of the target concepts. As this process invites occasions for their active inquiry, PMTs must make a host of design decisions for a variety of reasons; they draw on a range of conceptual, social, and material resources to mediate them.

We also appealed to Schön's (1992) notion of "knowing in action" (p. 2) in order to characterize and organize the resources that mediate participants' design decisions. Within a design setting, Schön considers knowledge to be in action as "the designer sees what is 'there'…, draws in relation to it, and sees what he/she has drawn, thereby informing further designing" (p. 5). This process of seeing-drawing-seeing is what Schön means by the phrase "designing as a reflective conversation with ... materials" (p. 5) It is this kind of conversation that is critical to Papert's (1980) constructionism, where conversations with artifacts are seen as essential for motivating and facilitating the construction of new knowledge (Ackermann, n.d.). It is also one that permits an analysis of these conversations to move beyond "static, explicit and objective" (Scheiner et al., 2019, p. 161) conceptions of teacher knowledge to recognize the dynamic, blended, and transformative (Scheiner, 2015) nature of knowing. Indeed, our use of "pedagogical/conceptual" is meant to acknowledge the inherent interplay between these domains of knowledge in a way that is consistent with Scheiner's framing. We, therefore, locate this learning by design approach to the invention of manipulatives at the "interplay between theory and practice, between constraints and tradeoffs, between designer and materials, and between designer and audience" (Mishra & Koehler, 2006, p. 1035).

Practice Phase

During the practice phase of their project, the PMTs used their designed manipulatives in a problem-solving interview with a target learner. In order to explore the mediating role of those tools in these teaching situations, we took a sociocultural perspective and grounded this work in the notion of *mediated activity*, derived from Vygotsky (1978) and advanced as instrumented activity by Verillon and Rabardel (1995). In terms of instrumented activity, an artifact is a material object that becomes an instrument (e.g., tool, sign) for the subject (e.g., actor, learner, teacher) when the subject has integrated it with their activity. Thus, an instrument is a psychological construct (as opposed to a material one) that "results from the establishment, by the subject, of an instrumental relation with an artifact" (p. 85). What the distinction between artifacts and instruments is meant to reveal is the possible range of actions a user might take with an artifact and what those actions might implicate about the user's knowledge. For our purposes, we are specifically interested in PMTs' pedagogical and conceptual knowledge and how their practice is mediated by such knowledge as it is intentionally embedded in their designed artifacts. As we analyze a PMT's use of their tool in practice, we use the term *embedding* to connote an intentional design decision that embeds a PMT's pedagogical and/or conceptual (i.e., mathematical) knowledge into a design element of their tool.

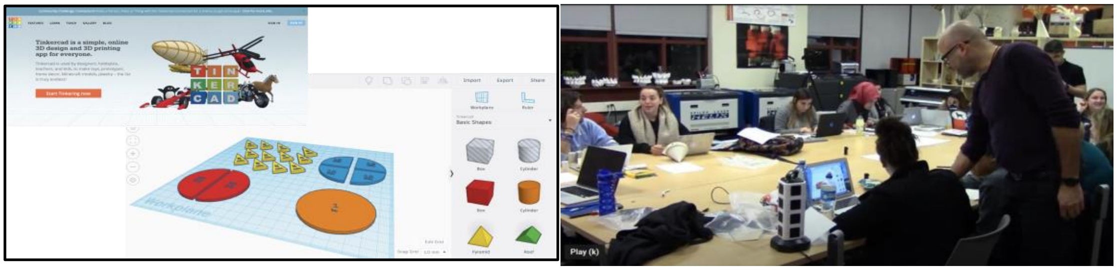
As an example, a PMT named "Moira" designed a fraction tool with a variety of fractional pieces of a whole. She was concerned that if each piece had its own unique color, that might "take away reasoning from children. If a student believes that a yellow ring represents sixths, they will immediately reach for yellow the second that they hear sixths." By giving the pieces the same color and leaving them "unmarked," she intended for children to construct their own meanings for each of the fractional pieces. Thus, we say that pedagogical/conceptual knowledge mediated this design decision and refer to the corresponding design element as an *embedding* of that knowledge. In addition, when we infer from a PMT's use of their manipulative in a teaching situation that the tool served as a resource for (e.g., a reminder of) pedagogical and/or conceptual knowledge embedded in the tool, we will refer to that as an *anchoring phenomenon*, as in this instance: "Moira's fraction tool served as an anchor for her attention to the pedagogical practice of implementing tasks that promote mathematical reasoning.

The Curricular Context and Experience

This study is part of a larger project that has been testing and refining the hypothesis that a pedagogically genuine, open-ended, and iterative design experience centered on the *Making* of a mathematical manipulative would be formative for the development of PMTs' inquiry-oriented pedagogy. Data collection for the larger project took place across two semesters of a graduate-level specialized mathematics course for prospective teachers of elementary mathematics at a mid-sized university in the northeastern United States. Forty students participated in the study. Thirty-four students worked alone, and the remaining six worked in pairs. Situated in an instructional context in which the teacher educators of those courses modeled an inquiry-oriented pedagogy, the course engaged students in a *Making* experience defined by the following task: "The purpose of this project is for you to 3D design and print an original physical tool (or 'manipulative') that can be used to teach a mathematical idea, along with corresponding tasks to be completed by a learner using the tool." The PMTs learned to use the Tinkercad (Autodesk, Inc., 2020; see Figure 1, left) digital modeling platform to design their manipulatives. They worked on their designs in in-class design sessions during three or four of the 90-minute weekly class meetings. These sessions were deliberately held in a design lab (Figure 1, right), so that the PMTs’ designing would be more inspired in an environment intentionally configured to accommodate the kind of immersive, collaborative social space that nourishes it.

Figure 1

The Tinkercad design environment (left) and the design setting (right).

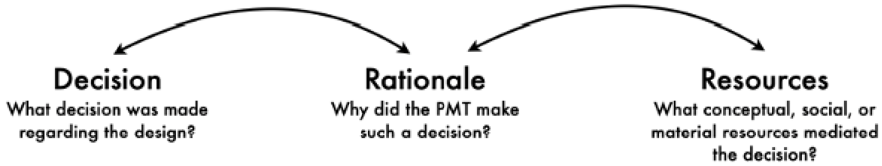


Methods

We took exploratory case study approaches (Yin, 2009) to this research. To address our first question, we took that approach in order to understand PMTs’ design activity by taking the three elements of each of their design decisions as the unit of analysis (see Figure 2): the decision itself, a rationale for making the decision, and the resources that mediated the decision making. The manipulative’s design, transcripts of video-recorded in-class design sessions, and four written project components formed the data corpus: 1) a “Math Autobiography” that calls on students to reflect on their experiences as a student of mathematics and consider how those experiences might inform their future work as mathematics teachers; 2) an “Idea Assignment” that describes PMTs’ initial thoughts about a manipulative they want to create; 3) a “Project Rationale,” which is an account of how their design reflects an understanding of what it means to know and learn mathematics; and 4) a “Final Paper/Reflection” that presents findings from a “Getting to Know You” interview and problem-solving interviews conducted by the PMTs with their tool and an elementary-age target student. We then took a grounded theory (Corbin & Strauss,

Figure 2

The 3 elements of a design decision



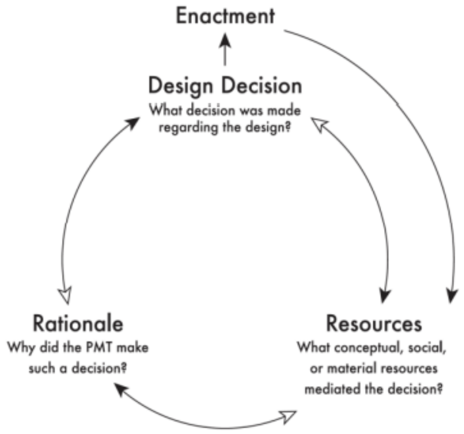
2008) approach to analyze the data. We began by analyzing the components of each design case and generating codes that characterize the design decisions and their mediating resources as they were revealed in the PMTs' written works and in the transcripts. As a reliability check on our analytic scheme, three additional researchers on the project individually analyzed the same design case, and then we convened to refine the coding process. This was followed by the constant comparison of data to ensure coherence across codes.

In order to address our second question, we took the exploratory case study approach to determine what connections could be made between pedagogical and conceptual rationales for PMTs' design decisions and how those designs mediated the pedagogical moves they made in enactment. Video recordings of their problem-solving interviews were added to the data corpus, and again we took a grounded theory (Corbin & Strauss, 2008) approach to analyze the data. We did so by taking the unit of analysis as instances in PMTs' teaching when the use of their manipulative implicated the pedagogical and/or conceptual knowledge underlying their design rationales. The locus of these particular research efforts among the broader research project is depicted as the arrow from "Design Decision" to "Enactment" in Figure 3.

We began by analyzing the written and video components of one PMT's design case to identify instances in their teaching from which we could infer that the PMT leveraged a particular *embedding* of a design decision in their manipulative to enact a teaching move that was consistent with aspects of their purported pedagogy, which they shared in the written artifacts of their Maker projects. These inferences constitute our conjectures that their designed manipulative served as an anchor for the pedagogical/conceptual knowledge they had been constructing in the course. We generated codes for this design case to characterize connections between *embeddings* of design decisions and their mediating role in the PMTs' teaching. Next, we identified additional instances of *anchoring* in other design cases. The analysis involved the constant comparison of data to ensure coherence is maintained across the generated codes and to get a good sense of the variety of ways in which affordances of the designed manipulatives that were either intended (those that PMTs intended to embed in their tool) or unintended (those that PMTs had not intended but realized in practice) could be leveraged to support a PMT's pedagogy.

Figure 3

Conceptual resources inform rationales for design decisions and may also be evoked in enactment.



Note: Open arrows acknowledge that feedback is reciprocally informing.

Results

Here we present excerpts of the exploratory case studies of three PMTs, "Roda," "Kerina," and

"Anyango." We chose these PMTs for these analyses for two reasons. First, their written work expressed the greatest number of design decisions among the thirty-four projects we analyzed. Second, there were instances of *anchoring phenomena* in their problem-solving interviews that could be traced back to design affordances whose rationales they had explicitly linked in their written work to their pedagogical and/or conceptual knowledge. As a result, their cases enabled us to identify and illuminate exemplars of the potential connections that can be made between the pedagogical and conceptual resources PMTs bring to their design decisions and how those designs mediated the pedagogical moves they made in practice.

We begin the presentation of these results by describing how the PMTs' rationales and resources mediated their design decisions as they made their manipulatives. We follow this with analyses of how their designed manipulatives mediated their teaching in a problem-solving interview setting that we regard as an approximation of practice (Grossman et al., 2009). We propose that the findings from these analyses identify instances of teaching mediated by a design embedding that served as an anchor for PMTs' pedagogical and/or conceptual attention.

Reasoning About the Unit Whole

Roda stated that her 5th-grade student was struggling with ordering decimals. Thus, she started "thinking about a tool that can help him build a conceptual understanding of how decimal numbers are constructed" (Roda's "Project Rationale" paper, p. 9). During their informal interview, Roda realized that her student likes snakes, so she decided to design a tool that she thought would honor his interest and engage his attention. Her "Decimal Snake" was designed to respond to his struggles by teaching him about decimals and decimal comparison (see Figure 4 below).

Figure 4

Roda's "Decimal Snake"

Roda's "Decimal Snake" consists of ten connected pieces. Each of these pieces is equally partitioned into ten parts. Thus, the decimal snake can be used to represent tenths of tenths, or hundredths of a whole, that is, any value between 0.01 and 1 to two decimal places. These design features are Roda's *embeddings* of the concepts of the whole and its decimal parts. Also, because Roda believed "colors have an impact on our thinking and perceptions… [because] our brains use color to recognize patterns and memory" (Roda's "Project Idea" paper, p. 2), she chose to give her tool just a single (cream) color. The pedagogical rationale mediating this design decision was that giving the tool multiple colors could influence her student's thinking. She, thus, intended for the student himself to "assign different colors [using marker pens] for each place value … to see the difference." Roda's design decision is a pedagogical one that allows the student to assign their own meanings, thereby promoting agentive self-exploration and discovery.

In the course of the interview, Roda challenges the child to compare 5.5 and 5.47. The child responds, "5.47 is 5 and 47 hundredths because it's 3 hundredths away from 5 and 5 tenths" (Roda's problem-solving interview, 10:36). Because Roda is interested in how her tool can support the child's reasoning, she challenges him to "Use the tool to show me?" (10:44). In the sixty seconds that follow, we witness the child struggling to locate 5.5 and 5.47 on the tool. Using the marker pen, he finally locates 5.5 at 0.55 and 5.47 at 0.47.

Figure 5

The child locates 5.5 and 5.47 on the decimal snake.

Ideally, these markings would only be possible if the entire snake represented 1. Given that several minutes earlier the child established that the entire snake is the "whole" (1:27) and that each piece of the snake is one tenth of a whole, we infer from his solution – locating 5.5 at 0.55 – that he had unintentionally designated each piece of the snake as 1 (as opposed to 0.01) and each partition of a piece as 0.1 (as opposed to 0.01). In doing so, he changed his designation of the entire snake from the whole (1) to 10, and consequently, each piece of the snake now represented 1. Thus, 5.5 would be presented as the 5th partition of the 5th piece.

Roda's next move aimed to help the child identify and resolve this confusion. When she asks him to "Show me one tenth" (12:22), he points to one of the tenth pieces. When she asks for "two tenths" (12:24), he points to the second piece. Then she asks, "Where is 5 and 5 tenths?" (12:35) And in doing so, she perturbed his thinking and provoked disequilibrium. Soon thereafter, he resolves it and declares, "Oh, wait! This [entire snake] is one whole! 5 and 5 tenths, you can't even make it out of the snake!" (12:46) In response to this unanticipated move in the child's activity, Roda leverages an affordance of her tool – namely that each piece of the snake could represent either a tenth of a whole or one of ten wholes – and she exploits it to support new ways of thinking for the child as he resolves his confusion about the representational capacities of the tool:

Roda: You need how many snakes to make 5.5?

Child: You need 5– No, 6 snakes!

Roda: How can we compare [*5.5 and 5.47*] using 1 snake? Is that possible?

Child: We can pretend that each piece is one snake. (12:50 – 13:15)

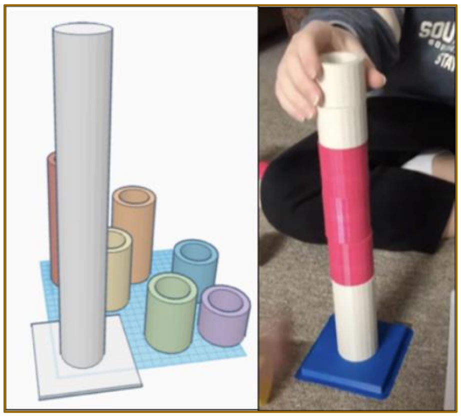
In this instance, Roda leverages the *embedding* of a conceptually resourced design decision that enabled the snake's user to engage in conversations about the unit whole. Specifically, she leveraged a design decision that allows for flexibility in naming the unit whole in relation to the snake and its pieces. And her rationale for leveraging that affordance was a pedagogical one. Rather than correct the child's interpretation, she sought to help him reason through his interpretations in order to resolve the confusion himself. In this respect, the tool's capacity for flexible interpretations of quantities (a conceptually resourced design decision) served as an anchor for pedagogical knowledge about the value of revealing student thinking and posing purposeful questions to advance their mathematical reasoning. Worth noting, Roda did not plan for this conversation to be about the unit whole, nor had she anticipated it. Regardless, her tool-mediated activity made it possible to do so.

Generating a Space of Inquiry

Kerina designed a fraction tool to invoke conversations about the meaning of a fraction's denominator (see Figure 5). She states that because "students often get confused when they see fractions with different denominators. This manipulative helps to show students that the concept of different denominators does not have to be as confusing as it is presented inside the classroom" (Kerina's "Project Rationale" paper, p. 3). Kerina's tool features "a variety of rings which each represent different fractions (from 1/2 to 1/8) that are scaled in relation to the pedestal [whole] that they go on top of." Each set of like fraction pieces is a "different color, so it's easy to determine which pieces are the same size" (p. 2). Kerina's imagined utilization scheme was mediated by conceptual and pedagogical intentions: when fraction pieces are stacked on the pedestal, the tool provides feedback to the child that they can use to determine whether that combination of fractions is equivalent to a whole.

Figure 6

Kerina's Fraction Tool



In terms of the role of aesthetics, Kerina decided that "each one of my fraction pieces is a different color, so it's easy to determine which pieces are the same size." In this way, if a child wanted to determine what fraction of a whole is represented by a pink piece, for example, they would make that determination by seeing how many pink pieces it takes to "fill" one pedestal. If 6 pink pieces fit on a pedestal, then each pink piece would represent ⅙. This finding would give meaning to the 6 in the denominator of fractions of the form *n*/6. As she designed her manipulative, Kerina was mindful that students tend to struggle with symbolic representations of fractions, particularly in the context of adding fractions and "finding least common denominators." As an alternative, she proposed that "students' brains will work in more creative ways than we can anticipate." Accordingly, she wanted to design a tool that would accommodate such diversity and enable students to "visualize" concepts and avoid the "frustration" that purely symbolic approaches to fractions often cause. With these intentions in mind, Kerina embeds a particularly salient feature of her pedagogy in the design of her tool that is made evident in a task she wrote that challenges a child to use the tool to, "Find three different ways to make a whole." Operating in tandem with a tool that requires its users to construct their own meanings for each of its pieces, the task generated a space (Stroup et al., 2004) for the child's active, creative, and playful inquiry and insight into fraction meanings and relationships. Indeed, Kerina designed her tool for such an imagined utilization scheme in which the child, at least initially, uses trial and error to stack different pieces onto the pedestal and then "see how much space is left" before adding on more pieces to make the whole. These accomplishments would be seen as groundings (Nathan, 2014) for connections she would subsequently help the child make as they learned the symbolic representations of their tool-based activity.

In practice, we observed Kerina's commitment to her design intentions. At one point, when she posed her "Find three ways" task, the child selected pieces of the same size to place on the pedestal in order to form a whole. Kerina noticed this strategy and asked the child to, "Try to use ones that have different denominators." Note her use of "different denominators" as opposed to "different sizes," even though she's referencing physical objects. In doing so, she is cultivating a connection between physical and symbolic representations of fractions. At the same time, it's also important to note that Kerina had written the symbolic names of each fraction piece on their interior where they could be concealed from the child's view. Thus, she seems to have a trajectory in mind for the meaningful development of fraction proficiency from physical to symbolic representations of collections of different unit fractions. Her tool and tasks anchored pedagogical and conceptual knowledge that mediated her response to the child's initial activity at that moment as she supported his construction of procedural fluency on a foundation of conceptual understanding. Specifically, design elements of her tool embedded conceptual knowledge relevant to that trajectory (e.g., a "complete" stack of pieces represents a sum of unit fractions equal to 1), and design elements of both the tool and the task embed pedagogical knowledge about the value of enabling multiple solution strategies in order to generate a space for open and productive inquiry.

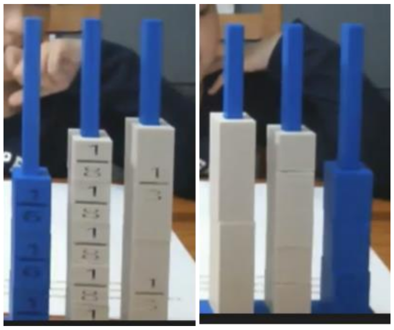
Noticing in Action

Anyango conceived of her design idea in response to the needs of a child she had worked with. She explained, "The student I am working with said she enjoys fractions and I'm hoping to make something really cool to help her gain another level of understanding." Anyango, thus, decided to design a tool that she believed could "help … [her] students visualize and deepen their understanding as they explore fraction relationships." She provided rationales for her student-centered design decision as she hoped to extend her student's current thinking about fractions.

Anyango's design is "a 3D version of fraction strips [see Figure 6]. Each strip was made to be a rectangular/square piece that slides into individual pegs… [the] fraction blocks stack vertically... to indicate height as value and amount." With several fractions mounted on a single "platform with the 1 (whole) always being visible … the student could begin to grasp how all the smaller parts can equate and compare to the whole" (Anyango's "Final Paper," p. 1). Here, technological knowledge, the mathematics of fractions, and a responsive pedagogy (e.g., Smith et al., 2016) served as resources that mediated these and other design decisions that embed fraction values and concepts into the tool. In terms of aesthetics, Anyango stated that, "The colors didn't matter much … Giving each fraction block its own color would have been aesthetically pleasing, but it did not affect how the manipulative worked" (pp. 1-2). To Anyango, colors do not have any pedagogical values as they only play a purely "aesthetic" role (p. 2).

Figure 7

Anyango's Fraction Pedestals



Note: Fraction names are embedded on just one face of each fraction piece.

In practice, Anyango posed the following task to her child: *Jack and his two friends each had the same size pizzas for lunch. Jack ate 5/8 of his pizza. Judy ate 2/3 of her pizza. And Sam ate 3/6 of his pizza. Who ate the most pizza? Who ate the least?* The child responded by stacking five one-eighth pieces, two one-third pieces, and three one-sixth pieces, each on their own pedestal with their labels facing her (Figure 6, right). Anyango's intention was for the child to compare "heights as amount" and identify the tallest as the one "who ate the most," and shortest as the one "who ate the least." When she asked the child, "Who ate the most?" the child attended exclusively to the symbolic representations engraved on each of the pegs and concluded that "It's Jack" (represented by the 5/8 piece), saying that, "5 out of 8 is the biggest of all of them … 2 out of 3 is smaller and 3 out of 6 is … kind of small." When Anyango asked the child to justify her answer, she explained, "The top is two and the bottom is three." We inferred from this response that the child was basing her comparisons on interpretations of fractions as two separate whole numbers. According to this way of thinking, 5/8 is greater than 2/3.

We interpret Anyango's next move as a noticing one (Sherin et al., 2011) as she leveraged her pedagogical knowledge about the efficacy of interpreting and attending to students' thinking:

Anyango: If I turn this [pedestal] around [Figure 6, left, such that the child's gaze can no longer be restricted to the fraction labels on the pieces], who ate the most?

Child: *<Pointing to Judy's stack of two one-third pieces:>* This one.

Anyango: Who has the least?

Child: <Pointing to Sam's stack of three sixth-pieces:> This one. (26:36 – 26:47)

What we find remarkable is that while Anyango made the intentional design decision to label each of her pieces, this "flipping" move leveraged an unintentional design affordance, that the opposite face of each piece is *not* labeled. In this regard, we suggest that Anyango's tool served as an anchor for a pedagogical knowing in action mediated by that affordance. Translating Schön's (1992) concept of knowing-in-action as a noticing-in-action, we suggest that in this instance, Anyango sees what is there, makes a move in relation to it, and sees what that move accomplishes, thereby informing her next steps. In those next steps, she returns the tool to its initial, label-facing orientation so that she can connect the physical representation of the amount to the symbolic one, and asks the child, "Who ate the most?" "Judy," she says with a smile, as she points to Judy's stack of fraction pieces.

Discussion

This work set out to explore teacher learning at the interface between theory and practice by discerning the pedagogical/conceptual knowledge that prospective teachers of elementary mathematics bring to their design activity and whether connections can be made from that knowledge as it is constructed in teacher preparation into teachers' practice. The following questions framed the inquiry: "*As prospective teachers Make new manipulatives for mathematics teaching and learning, (1) What is the nature of the resources and rationales they bring to their design decisions and how do these intersect to mediate their decision making? (2) Can connections be made between pedagogical/conceptual resources for their design decisions and how those designs mediate the pedagogical moves they make in practice?*" We pursued this inquiry by analyzing prospective teachers' design activity followed by approximations of their practice in order to identify instances in their teaching when their manipulative served them as a mediating anchor for pedagogical and/or conceptual knowledge acquired in teacher preparation coursework, resourced in their design decisions, and embedded in their designs.

Findings from this work revealed that a host of new possibilities are afforded to teachers at the intersection of digital design and fabrication technologies, learner-centered design practices, and inquiry orientations to mathematics teaching and learning. In particular, we hypothesized that a pedagogically genuine design experience would be formative for the development of an inquiry-oriented pedagogy that is responsive to the particular needs and interests of actual learners. The quality and diversity of design decisions made by the prospective teachers, as well as the breadth of conceptual, pedagogical, social, cultural, and experiential resources, brought to bear upon them, speaks to the generative power of the open-ended and iterative design experience in terms of the agency prospective teachers assumed in their design activity and the wealth of knowledge they leveraged to mediate it. Indeed, the scope of resources that were evoked and brought to bear upon these decisions betrays deficit framings of elementary mathematics teachers (Association of Mathematics Teacher Educators, 2009) and warrants their repositioning as teachers with expertise with all the authority of agents of curricular and pedagogical reform. Moreover, by extending our inquiry from teacher preparation into practice, our findings of instances of *anchoring phenomena* suggest that constructionist *Making* experiences have the potential to yield material epistemic scaffolding (e.g., in physical manipulative form) that supports teachers in practice as they aim to maintain their commitments to the models of knowing and learning they constructed in teacher preparation. These findings have implications for theory as well, for they demonstrate the analytic value of our design, rationale, resource, and practice (DRR-P) framework for revealing the benefits and opportunities of these experiences in teacher preparation.

Conclusion

This work has demonstrated the formative value of immersing prospective teachers in a communal design environment of collective social *Making* and tasking them with a pedagogically genuine design experience centered on the *Making* of an original physical manipulative for mathematics teaching and learning. Its findings contribute to research on teachers learning by design while also generating new opportunities for research that moves the field forward regarding the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation. Future research could more closely explore the features of productive design environments for teachers' making, the teacher educator's role in designing and facilitating these experiences, and the subsequent in-service instruction of teachers who participated in these experiences during teacher preparation.

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Corresponding Author Contact Information:

Author name: Denish Akuom

Department: Department of Mathematics

University, Country: Montclair State University, USA

Email: akuomd1@montclair.edu

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